

Quantum Interrogation and the Safer X-ray

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We investigate quantum interrogation techniques which allow imaging information about semi-transparent objects to be obtained with lower absorption rates than standard classical methods. We show that a gain proportional to $\log N$ can be obtained when searching for defects in an array of N pixels, if it is known that at most M of the pixels can have transparencies different from a predetermined theoretical value. A logarithmic gain can also be obtained when searching for infrequently occurring large structures in arrays.

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I. INTRODUCTION

The well-known Elitzur-Vaidman quantum interrogation scheme [1] uses interferometry in a way which sometimes allows photo-sensitive objects to be detected without absorbing a photon. In EV's original example, starting from a mixed supply of “good” bombs that will explode if a single photon hits the trigger and of dud bombs which reflect photons without exploding, a guaranteed supply of unexploded good bombs can be produced. EV's probabilistic scheme was significantly refined by Kwiat et al. [2], who showed that the probability of absorption-free detection of a photo-sensitive object can be made arbitrarily close to one by a quantum Zeno interferometry technique.

These schemes both distinguish perfectly opaque from perfectly transparent (or, in the original formulation, perfectly reflecting) objects. They inspired the hope that effectively absorption-free quantum interrogation methods could also be used to discriminate semi-transparent objects of different transparencies. Such absorption-free discrimination, if possible, would be of great practical value. It could, for example, allow a “safe X-ray”, in which images of a patient's internal structure could be obtained with arbitrarily low absorption of harmful X-rays.

In fact, though, the Kwiat et al. quantum Zeno technique does not allow absorption-free discrimination between partially opaque objects [3]. More generally, Mitchison and Massar have shown [4] that there is a non-zero bound on the probabilities of absorption involved in any scheme discriminating between two imperfectly opaque objects.

The Mitchison-Massar bound might be taken as dampening hope of finding any seriously valuable imaging applications of quantum interrogation other than discriminating a partially opaque and a completely transparent object. However, such pessimism would be un-

justified. We describe here two quantum interrogation schemes which show that quantum information can indeed be used to extract imaging information about semi-transparent objects with significantly reduced absorption rates. Though the safe X-ray is impossible, a somewhat safer X-ray is viable, at least in theory.

We represent the object to be imaged as an array of N pixels, labelled by i from 1 to N , with transparencies α_i for the relevant radiation. That is, the interaction between a photon and the i -th pixel is described by

$$|\gamma_i^{\text{in}}\rangle|i\rangle \rightarrow \alpha_i|\gamma_i^{\text{out}}\rangle|i\rangle + \beta_i|i^*\rangle, \quad (1)$$

where $|\gamma_i^{\text{in}}\rangle$ and $|\gamma_i^{\text{out}}\rangle$ describe incoming and outgoing photons in a beam passing through the i -th pixel, $|i\rangle$ is the initial state of the i -th pixel, $|i^*\rangle$ is the excited state of the i -th pixel after absorbing a photon, and $|\alpha_i|^2 + |\beta_i|^2 = 1$, with $0 \leq |\alpha_i| \leq 1$.

Our aim is to extract useful imaging information about the pixel array transparencies while minimizing the expected number of photons absorbed in the array. Specifically, we are interested in quantum methods which do better in this respect than standard X-ray imaging. A standard X-ray image is effectively obtained by firing a number of photons separately through each of a number of different pixels, counting the number of outgoing photons, and statistically inferring information about the individual pixel transparencies or some joint functions thereof. This “classical” method makes no essential use of the fact that quantum information is being generated in the imaging process. Indeed, it makes no use of quantum theory at all, beyond the fact that photons are quantised. Effectively the same method could be used with classical radiation, assuming some finite bound on the sensitivity with which the amplitude of the outgoing radiation can be measured, and indeed this is how most standard imaging schemes work in practice.

From the perspective of quantum information theory, however, the classical method seems unlikely to be optimal for general problems. The pixel array behaves like

a non-unitary oracle in its action on incoming quantum states. It is well known in quantum computing that interrogating a unitary oracle by quantum superposition states and processing the outputs can be far more efficient than invoking it only on the basis states which define its action. It should not be surprising that this turns out also to be true of non-unitary oracles. In fact, as we show here, it is true in simple examples which arise naturally as imaging problems.

II. TESTING AN ARRAY FOR DEFECTS

Consider the following situation. We know that the transparency of pixel i should be α_i , if nothing untoward has occurred. However, we are concerned that there may be a small number of pixels whose transparency is in fact significantly different from the theoretical value, and we wish to check on this. To be precise, we have some fixed parameter ϵ , and know that up to M of the pixels may have an altered transparency α'_i , where $|\alpha'_i - \alpha_i| \geq \epsilon$, while the rest of the transparencies are unaltered. We wish to be statistically certain, to within some prescribed degree of confidence $(1 - \delta)$, whether or not there are any altered transparency pixels. If there are, we wish to identify them.

This problem would arise, for example, for pixel arrays manufactured by a generally reliable process that occasionally suffers from some specific defect on individual pixels. It is also an idealisation of the imaging problem that arises when we have previously taken an X-ray of a (then) healthy patient and now want to know whether any significant change has occurred, if we may assume that any changes will be confined to a small part of the image.

In the following discussion we take M , δ and ϵ to be fixed, with N varying. We are interested in the behaviour for large N ; in particular we assume N is large compared to M^2 , ϵ^{-2} and δ^{-1} . We also assume the α_i are generally not close to 0 or 1.

The classical method requires estimating the transparency of each pixel separately. This is done by firing photons through each pixel and ensuring that the transparencies can all be identified, either as the expected α_i or as differing by at least ϵ , with confidence $(1 - \delta)$ that none of the N identifications is erroneous. A classically obtained image giving this degree of confidence separately for each of the individual pixel identification requires $O(\epsilon^{-2}N)$ absorptions. However, as N grows, the chance that at least one of the N pixel transparencies will be misestimated by more than ϵ grows logarithmically in N . Allowing for this, we see that the classical solution to the problem defined requires an absorption rate of $O(\epsilon^{-2}N \log(N))$.

Consider now the following quantum method. Each incoming photon is split into an equally weighted super-

position of states incident on each of the pixels. That is:

$$|\gamma_{\text{in}}\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{i=1}^N |\gamma_{\text{in}}^i\rangle. \quad (2)$$

Let the actual transparency of pixel i be β_i . After passing through the pixels, the photon, if it was not absorbed, is in the state

$$\frac{C}{\sqrt{N}} \sum_{i=1}^N \beta_i |\gamma_{\text{out}}^i\rangle, \quad (3)$$

where C is a normalisation constant and $|\gamma_{\text{out}}^i\rangle$ are the states of photons emerging on the far side of pixel i .

We now need to know whether the photon was absorbed or not. If the absorptions themselves are not easily detected, this can in principle be tested by carrying out a von Neumann measurement of the photon number of the outgoing beam state. Assuming no absorption, we now carry out a further von Neumann measurement, measuring the projection P_ψ onto the state

$$|\psi\rangle = C' \sum_{i=1}^N \alpha_i |\gamma_{\text{out}}^i\rangle. \quad (4)$$

If we obtain eigenvalue 0, we know that the normalised outgoing state was not $|\psi\rangle$, and hence that not all the pixel transparencies can have the theoretical value. In this case, we apply a second measurement, simultaneously measuring the projections onto the states $|\gamma_{\text{out}}^i\rangle$ for i from 1 to N . In other words, we look to see which of the outgoing beams we find a photon in. We take the answer i to indicate that pixel i is defective.

Reducing the probability of a false negative — a defective pixel array producing eigenvalue 1 throughout the sequence of tests — to δ requires $O(\log(\delta^{-1})N\epsilon^{-2})$ such tests. and hence (given our assumptions about the α_i) this order of absorptions. Comparing the classical technique, and considering only the N -dependence for large N (since the other parameters are fixed), we find a reduction of a factor of $O(\log N)$ in the absorption rate.

When the defect test is positive, the probability of misidentifying a good pixel as defective when we measure the outgoing beam is $O(M/N)$. If this level of certainty is acceptable, a pixel thus identified as defective can be excluded from later tests. Up to M defective pixels can then be identified as defective, with the probability of any erroneous identification being $O(M^2/N)$. If these certainty levels are insufficient, we may require more than one positive result to identify a defective pixel. For any fixed level of certainty, however, the overall gain over the classical method remains proportional to $\log N$.

These calculations assume that the theoretical values of the α_i are determined with absolute precision. In the realistic case in which these values are themselves subject to errors, of sizes bounded by ϵ' , the conclusions hold so long as $\epsilon' \ll (M\epsilon/N)$.

III. SEARCHING FOR INFREQUENTLY OCCURRING LARGE STRUCTURES

We again consider an array of N pixels, and suppose now that we are looking for a particular type of rarely occurring image. We model this as follows. Our aim is to decide, with confidence $(1 - \delta)$, whether the array has a particular pattern $\{\alpha_i^0\}$ of transparencies. We suppose that the prior probability of this pattern occurring is some small but non-zero number p , and that with probability $(1 - p)$ the pixel transparencies are randomly drawn from identical independent probability distributions. To be definite, we suppose that in this latter case the pixel transparencies are uniformly distributed in the complex unit disc. We suppose also that $\exp(-\sqrt{N}) \ll \delta p$.

We neglect constant factors throughout this section, considering only the degree of dependence on the parameters N, δ, p .

Classically, we can only approach this problem by obtaining statistical estimates of some or all of the $|\alpha_i|$, and comparing the estimates to $|\alpha_i^0|$. Estimating the $|\alpha_i|$ on r pixels to within error ϵ requires $\approx r\epsilon^{-2}$ photon measurements and absorptions. Suppose that r such measurements produce the estimates $||\alpha_i| - |\alpha_i^0|| < \epsilon$. For this to give us the required confidence that we have found an example of the image requires that $\epsilon^r \approx \delta p$. Minimizing with respect to ϵ , we find this strategy requires $\approx -\log(\delta p)$ absorptions.

In a quantum approach to the problem, we proceed as in the previous example, preparing an equal superposition of photon states incident on the pixels and, if there is no absorption, testing whether the emerging state is $|\psi^0\rangle = C' \sum_{i=1}^N \alpha_i^0 |\gamma_{\text{out}}^i\rangle$. In the case where the pixel transparencies were randomly drawn from uniform distributions, the probability that they are such that the emerging state $|\psi\rangle$ obeys $|\langle \psi_0 | \psi \rangle|^2 > \frac{1}{\sqrt{N}}$ is $\approx \exp(-\sqrt{N})$, which is negligible in our calculations. So we may assume that a randomly drawn array will have transparencies such that $|\langle \psi_0 | \psi \rangle|^2 < \frac{1}{\sqrt{N}}$. We can confidently conclude that the image sought is present after x successful (and no unsuccessful) tests, where $(\frac{1}{\sqrt{N}})^x \approx \delta p$, so that $x \approx \max(\frac{-\log(\delta p)}{\log N}, 1)$. As this represents the order of the number of absorptions required, we again find a logarithmic quantum advantage.

These arguments generalise to a search for an infrequently occurring image which may be any one of N

known possibilities whose transparencies define N orthogonal states $|\psi_i\rangle$.

IV. CONCLUSIONS

The methods we have described show that quantum interrogation can have useful advantages over standard classical methods for realistic large array imaging problems. It is perhaps worth noting that these methods apply equally well to time-dependent imaging problems. A single array in our model could, for example, represent a smaller array being repeatedly probed at a sequence of times.

A further example of a problem in which there is a logarithmic quantum advantage has been found by Massar et al. [5]. It would be good to have a general understanding of the range of problems in which there is a quantum advantage. It would also be useful to identify the advantage attainable by optimal techniques, both for the problems we have described and more generally. Here the bounds obtained in independent work by Massar et al. [5] may be of help.

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